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Efficiency at maximum power of Feynman's ratchet as a heat engine

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Abstract
The maximum power of Feynman's ratchet as a heat engine and the corresponding efficiency (η_*) are investigated by optimizing both the internal parameter and the external load. When a perfect ratchet device (no heat exchange between the ratchet and the pawl via kinetic energy) works between two thermal baths at temperatures $T_1 > T_2$, its efficiency at maximum power is found to be $\eta_* = \eta_C^2 \big/ [\eta_C - (1-\eta_C)\ln(1-\eta_C)]$, where $\eta_C \equiv 1 - T_2/T_1$. This efficiency is slightly higher than the value $1 - \sqrt{T_2/T_1}$ obtained by Curzon and Ahlborn (1975 *Am. J. Phys.* **43** 22) for macroscopic heat engines. It is also slightly larger than the result $\eta_{SS} \equiv 2\eta_C/(4-\eta_C)$ obtained by Schmiedl and Seifert (2008 *EPL* **81** 20003) for stochastic heat engines working at small temperature differences, while the evident deviation between η_* and η_{SS} appears at large temperature differences. For an imperfect ratchet device in which the heat exchange between the ratchet and the pawl via kinetic energy is non-vanishing, the efficiency at maximum power decreases with increase in the heat conductivity.

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As is well known, the Carnot efficiency $\eta_C \equiv 1 - T_2/T_1$ gives the upper bound for heat engines working between two thermal baths at temperatures $T_1 > T_2$. However, the engines at the Carnot efficiency cannot produce a power output because the Carnot cycle requires an infinitely slow process. The cycle should be speeded up to obtain a finite power. Curzon and Ahlborn [1] derived the efficiency, $\eta_{CA} \equiv 1 - \sqrt{T_2/T_1}$, at maximum power for macroscopic heat engines within the framework of finite-time thermodynamics. The same expression was also obtained from linear irreversible thermodynamics for perfectly coupled systems [2–4]. It is pointed out that the efficiency at maximum power might be different from η_{CA} for imperfectly coupled systems [2–4]. In a recent paper [5], Schmiedl and Seifert investigated cyclic Brownian heat engines working in a time-dependent harmonic potential and obtained

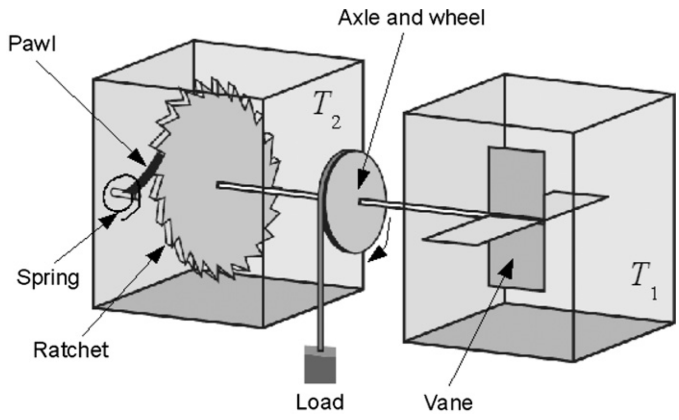

Figure 1. Feynman's ratchet device.

the efficiency at maximum power, $\eta_{SS} \equiv 2\eta_C/(4-\eta_C)$, within the framework of stochastic thermodynamics [6–8].

To illustrate the second law of thermodynamics, Feynman introduced an imaginary microscopic ratchet device in his famous lectures [9]. Following Feynman's spirit, many models were put forward, such as on-off ratchets [10], fluctuating potential ratchets [11, 12], temperature ratchets [13, 14], chiral ratchets [15–17], and so on, which have potential applications in biological motors. Thus it is significant to investigate the efficiency and power of these ratchets. Feynman's ratchet, as a parental model, attracts researchers' interests [18–21] in the nature of things. In particular, the efficiency at maximum power of Feynman's ratchet was obtained by optimizing the external load for a given internal parameter in [19–21]. However, there is still lack of a result by optimizing both the internal parameter and the external load of the ratchet device. In this paper, we will analytically derive this result.

Let us consider Feynman's ratchet device as shown in figure 1. It consists of a ratchet, a pawl and spring, vanes, two thermal baths at temperatures $T_1 > T_2$, an axle and wheel, and a load. For simplicity, we assume that the axle and wheel is a rigid and frictionless thermal insulator.

Now we follow Feynman's discussion [9]. In one-step forward motion, we must borrow an energy ϵ to overcome the elastic energy of spring and then lift the pawl. Assume that the wheel rotates at an angle θ per step and the load induces a torque Z, thus we also need an additional energy $Z\theta$. Then the total energy that we have to borrow is $\epsilon + Z\theta$. In fact, we can borrow it from the hot thermal bath in the form of heat. The rate to get this energy is

$$R_F = r_0\,\mathrm{e}^{-(\epsilon+Z\theta)/T_1}, \tag{1}$$

where r_0 is a constant with dimension s^{-1}, and the Boltzmann factor is taken to 1. In this process, the ratchet absorbs heat $\epsilon + Z\theta$ from the hot thermal bath. A part of this heat is transduced into work $Z\theta$, and the other is transferred as heat ϵ to the cold thermal bath through the interaction between the ratchet and the pawl.

Now let us consider a one-step backward motion. To make the wheel backward, we have to accumulate the energy ϵ to lift the pawl high enough so that the ratchet can slip. Here the rate to get this energy is

$$R_B = r_0\,\mathrm{e}^{-\epsilon/T_2}. \tag{2}$$

In the backward process, the load does work $Z\theta$. This energy and the accumulated energy ϵ are returned to the hot thermal bath in the form of heat.

In an infinitesimal time interval Δt, the net work done on the load by the system may be expressed as

$$W = (R_F - R_B)Z\theta\Delta t. \tag{3}$$

The net heat absorbed from the hot thermal bath via the potential energy [22, 23] may be expressed as:

$$Q_1^{\text{pot}} = (R_F - R_B)(\epsilon + Z\theta)\Delta t. \tag{4}$$

Since the ratchet contacts simultaneously with two thermal baths at different temperatures, there may exist a heat conduction from the hot thermal bath to the cold one via the kinetic energy [22, 23]. In time interval Δt, it can be expressed as

$$Q_1^{\text{kin}} = \sigma(T_1 - T_2)\Delta t, \tag{5}$$

where σ is the heat conductivity due to the heat exchange between the ratchet and the pawl via kinetic energy. The analysis by Parrondo and Español suggests that σ is inversely proportional to the masses of the ratchet and the pawl [22]. We first consider a perfect ratchet device in which the masses of the ratchet and the pawl are infinitely large relative to the gas molecules full in both thermal baths. In this case, σ and Q_1^{kin} are vanishing. Thus the efficiency can be defined as

$$\eta = W\big/Q_1^{\text{pot}} = Z\theta/(\epsilon + Z\theta). \tag{6}$$

The power is defined as

$$P = W/\Delta t = r_0 Z\theta[\mathrm{e}^{-(\epsilon+Z\theta)/T_1} - \mathrm{e}^{-\epsilon/T_2}]. \tag{7}$$

We find that P depends on the internal parameter ϵ and the external load Z. It is easy to tune the external load Z. In fact, ϵ can also be adjusted by changing the strength of the spring. We can optimize both ϵ and Z to achieve the maximum power. Before doing that, we introduce two dimensionless parameters $\varepsilon = \epsilon/T_2$ and $z = Z\theta/T_1$. Equations (6) and (7) are then respectively transformed into

$$\eta = \frac{z}{\varepsilon(1-\eta_C) + z}, \tag{8}$$

and

$$P = r_0 T_1 z[\mathrm{e}^{-\varepsilon(1-\eta_C)-z} - \mathrm{e}^{-\varepsilon}]. \tag{9}$$

Maximizing P with respect to ε and z, we have

$$\begin{cases}(1-\eta_C)\,\mathrm{e}^{\varepsilon\eta_C - z} = 1,\\ (1-z)\,\mathrm{e}^{\varepsilon\eta_C - z} = 1.\end{cases} \tag{10}$$

The solution to the above equation is

$$\begin{cases}z_* = \eta_C,\\ \varepsilon_* = 1 - \eta_C^{-1}\ln(1-\eta_C).\end{cases} \tag{11}$$

Substituting equation (11) into equation (8), we derive the efficiency at maximum power

$$\eta_* = \frac{\eta_C^2}{\eta_C - (1-\eta_C)\ln(1-\eta_C)}. \tag{12}$$

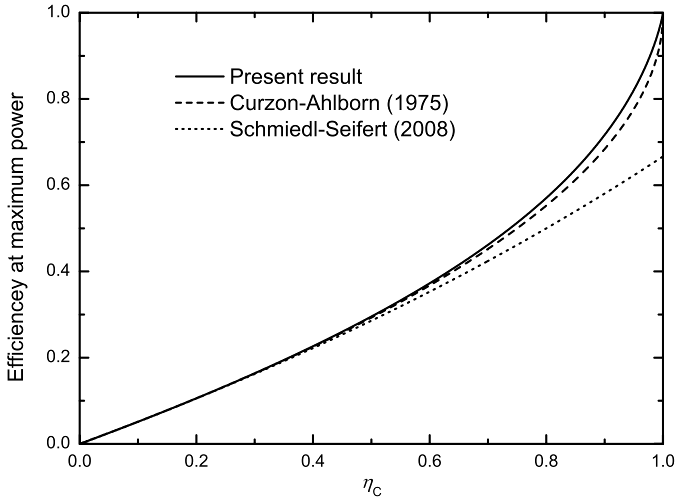

Figure 2. Comparison between typical results on the efficiency at maximum power obtained by different research groups.

This is the main result in the present paper. It is not hard to prove $\eta_{CA} < \eta_* < \eta_C$ through simple analysis. Similarly, substituting equation (11) into equation (9), we obtain the maximum power

$$P_* = \frac{r_0 T_1 \eta_C^2}{e(1-\eta_C)^{1-\eta_C^{-1}}}. \tag{13}$$

Now we compare the present result with two typical values η_{CA} and η_{SS} at various temperatures. Figure 2 shows the relation between the efficiency at maximum power and the relative temperature difference, $(T_1 - T_2)/T_1 = \eta_C$. We find that the present result (solid line) is slightly higher than the value $\eta_{CA} \equiv 1 - \sqrt{1-\eta_C}$ obtained by Curzon and Ahlborn (dash line) for macroscopic heat engines, and than the result $\eta_{SS} \equiv 2\eta_C/(4-\eta_C)$ obtained by Schmiedl and Seifert (dot line) for stochastic heat engines at a small relative temperature difference (for example, $\eta_C < 0.5$). The evident deviation between the present result and η_{SS} appears merely for $\eta_C > 0.5$.

Now we investigate the asymptotic behaviors of η_*, η_{CA} and η_{SS} at small relative temperature differences. Expanding them up to the third-order term of η_C, we have

$$\eta_* = \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \frac{7\eta_c^3}{96} + O\left(\eta_c^4\right), \tag{14}$$

$$\eta_{CA} = \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \frac{6\eta_c^3}{96} + O\left(\eta_c^4\right), \tag{15}$$

and

$$\eta_{SS} = \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \frac{3\eta_c^3}{96} + O\left(\eta_c^4\right). \tag{16}$$

These results deviate from each other at the third-order term of relative temperature difference, which suggests that a universal efficiency at maximum power, $\eta_c/2 + \eta_c^2/8$, should exist at small relative temperature differences.

Up to now, we have only discussed the ideal case that the masses of ratchet and pawl are infinitely large relative to the gas molecules full in both thermal baths such that σ and Q_1^{kin} are

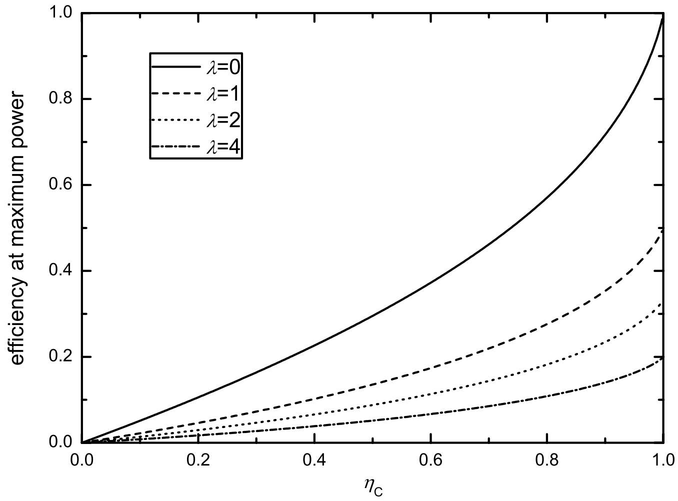

Figure 3. Efficiency at maximum power for different heat conductivities.

vanishing. In practice the masses are finite and the ratchet device is imperfect, so σ and Q_1^{kin} are non-vanishing. In this case, the net heat absorbed from the hot thermal bath is [19, 22, 23]

$$Q_1 = Q_1^{\text{pot}} + Q_1^{\text{kin}} = [(R_F - R_B)(\epsilon + Z\theta) + \sigma(T_1 - T_2)]\Delta t. \tag{17}$$

Correspondingly, the efficiency is modified as

$$\eta' = W/Q_1 = \frac{z}{\varepsilon(1-\eta_C) + z + \mathrm{e}\lambda\eta_c[\mathrm{e}^{-\varepsilon(1-\eta_C)-z} - \mathrm{e}^{-\varepsilon}]^{-1}}, \tag{18}$$

where $\lambda \equiv e\sigma/r_0$ is the reduced heat conductivity. The power P is independent of the heat conductivity σ, so equations (9)–(11) and (13) are unchanged.

Substituting equation (11) into (18), we obtain the efficiency at maximum power

$$\eta'_* = \frac{\eta_C^2}{\eta_C - (1-\eta_C)\ln(1-\eta_C) + \lambda\eta_C(1-\eta_C)^{1-\eta_C^{-1}}}. \tag{19}$$

It is obvious that $\eta'_* < \eta_*$ because $\lambda\eta_C(1-\eta_C)^{1-\eta_C^{-1}} > 0$. The efficiency at maximum power depends on the reduced heat conductivity λ. From figure 3, we see that it decreases with increase in the reduced heat conductivity.

In summary, we have investigated the efficiency at maximal power of Feynman's ratchet as a heat engine by optimizing both the internal parameter and the external load. We have analytically derived the efficiency at maximum power, equation (12), for the perfect ratchet device, which is slightly higher than the value obtained by Curzon and Ahlborn. The efficiency at maximum power for the imperfect ratchet device, equation (19), decreases with increase in the reduced heat conductivity. The present result for the perfect ratchet device deviates from the value obtained by Schmiedl and Seifert merely at large relative temperature differences.

Finally, we have to list a few open problems which should be addressed in the future work. (i) What is the underlying reason for the deviation between η_* and η_{SS} at large temperature differences? A possible conjecture is that there might not exist a universal formula at large relative temperature differences. (ii) Zhang *et al* held a different viewpoint on σ [21]. They argued that $\sigma = (R_F + R_B)/2$, which is independent of the masses of ratchet and pawl. By adopting their argument, one can derive the efficiency at maximum power to be $2\eta_C/\left[4-\eta_C-2\left(\eta_C^{-1}-1\right)\ln(1-\eta_C)\right]$ which is much smaller than η_{CA}, η_{SS}, and η_* for $0<\eta_C<1$. Whether σ does indeed depend on the rates R_F and R_B? (iii) Allahverdyan *et al* investigated a class of quantum heat engines consisting of two subsystems interacting with a work source and coupled to two separate baths at different temperatures [24]. They also found that the efficiency at maximum power of these quantum heat engines was slightly larger than η_{CA}. Is there any relation between their result and ours?

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